Hyperbolic Models of Homogeneous Two-Fluid Mixtures

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Abstract. One derives the governing equations and the Rankine - Hugoniot conditions for a mixture of two miscible fluids using an extended form of Hamilton's principle of least action. The Lagrangian is constructed as the difference between the kinetic energy and a potential depending on the relative velocity of components. To obtain the governing equations and the jump conditions one uses two reference frames related with the Lagrangian coordinates of each component. Under some hypotheses on flow properties one proves the hyperbolicity of the governing system for small relative velocity of phases.

Sommario. Le equazioni di governo e le condizioni di Rankine-Hugoniot sono derivate per una miscela di due fluidi miscibili usando una formulazione estesa del principio di minima azione di Hamilton. La Lagrangiana è costruita come differenza tra energia cinetica e potenziale dipendente dalla velocità relativa dei componenti. Per ottenere le equazioni di governo e le condizioni di salto si usano due sistemi di riferimento collegati alle coordinate Lagrangiane di ciascun componente. Nelle stesse ipotesi sulle propretà di flusso si prova l'iperbolicità del sistema per piccole velocità relative delle fasi.

Key words: Hamilton's principle, Hyperbolicity, Multiphase Flows.

1. Introduction

The theory of mixtures considers two different kinds of continua: *homogeneous* mixtures (each component occupies the whole volume of the physical space) and *heterogeneous* ones (each component occupies only a part of the mixture volume). In the second case, the geometrical parameters appear as the unknown variables: the volume concentration, sizes of dispersed particles etc. The problem, which is common for both types of media, is to describe two-velocity effects, that are responsible for the development of instability in mixtures, diffusion of components etc.

At least three approaches to the construction of two-fluid models are known. The most common one for studying of heterogeneous two-phase flows is the *averaging method* (Ishii [1], Nigmatulin [2] and others). Averaged equations of motion are obtained by applying an appropriate averaging operator to the balances laws of mass, energy etc., which are valid inside each phase. The main problem associated with this approach is the closure of the system obtained: the system contains more unknowns than equations. Different experimental and theoretical hypothesis are used for the closure. Not all of them give *well-posed* governing equations. For example, it was noted by many authors (Drew [3], Kraiko & Sternin [4], Stuhmiller [5] and others) that non-dissipative averaging governing equations of heterogeneous two-velocity media are *not hyperbolic* even for small relative velocity of the mixture components, when the equal pressures hypothesis in phases is used. This implies that the Cauchy problem for the corresponding equations of motion is *ill-posed*. The nonhyperbolicity was overcome later by many authors: Liapidevskii [6], Ransom & Hicks [7] (two-phase separated flow), Biesheuvel & van Wijngaarden [8] (bubbly liquids), Fabre *et al* [9] (slug flow) etc. The well-posedness of governing equations was achieved by using additional closure relations for averaged quantities, which are specific to the flow type. The hyperbolicity of one-dimensional models only was proved.

A different approach known as *Landau method of conservation laws* was initially used for constructing models of quantum liquids such as superfluid helium (Khalatnikov [10], Landau & Lifshits [11], Putterman [12]). The method consists in the following: the requirement of the fullfillment of balance laws of mass, energy etc., complemented by the Galilean relativity principle and the Gibbs thermodynamic identity fully determines the governing equations of motion. Recently this approach was applied to classical fluids (two-velocity hydrodynamics) by Dorovsky & Perepechko [13], Roberts & Loper [14], Shugrin [15]. The method does not take into account the geometrical characteristics of the mixture components: the volume concentrations, sizes of particles etc. In the non-dissipative case it gives also hyperbolic models (see, for instance, Khalatnikov [10], where sound velocities for superfluid helium are calculated).

Finally, a third approach called *variational method* is the most universal. Bedford & Drumheller [16], Berdichevsky [17], Geurst [18,19] have applied it for investigation of bubbly liquids flows. In particular, Geurst has proven in one-dimensional case the hyperbolicity of the governing equations for small relative velocity of phases.

These methods present three different approaches for description of complex media. At present, their common features and distinctions are not quite well understood.

We consider the variational approach to describe two-velocity effects in homogeneous mixtures. A physical example of such a flow is a mixture of two miscible fluids, or a mixture of two gases with quite different molecular weights.

In Section 2 we introduce an extended form of Hamilton's principle of least action. The Lagrangian of the system is chosen in a general form: it is the difference of the kinetic energy of the system, which depends obviously on the choice of a reference frame, and a thermodynamic potential, which is a Galilean invariant, conjugated to the internal energy with respect to the relative velocity of phases. If it does not depend on the relative velocity of components, we have a classical form of Hamilton's action for two-velocity systems (see, for instance, the article by Gouin [20], where the thermo-capillary mixtures were studied).

In Section 3 we get from the variational principle the governing equations and the Rankine-Hugoniot conditions for shocks. To obtain the desired relations, we used two reference frames related with Lagrangian coordinates of each component.

Conservation laws of the total energy and the total momentum are derived in Section 4. We show that, under some restrictions on the flow properties, the governing equations admit additional conservation laws in terms of the densities ρ_1, ρ_2 and the velocities $\mathbf{u}_1$, $\mathbf{u}_2$ of components. Without these restrictions the system seems not to be conservative. We extend the set of the unknown variables, considering the deformation gradients as the required quantities, and rewrite our system in a conservative form that gives additional set of possible jump conditions.

In Section 5, classification of strong discontinuities is done and some difficulties of the "right" choice of jump conditions are discussed.

We investigate in Section 6 the hyperbolicity of the governing system for small relative velocity of phases in multi-dimensional case. Under some hypotheses on flow properties, we reduce our system to Friedrichs' symmetric form and prove that convexity of the internal energy guarantees hyperbolicity of the governing equations.

As a convention, in the following we shall use asterisk "*" to denote *conjugate* (or *transpose*) mappings or covectors (line vectors). For any vector $\mathbf{a}$, $\mathbf{b}$ we shall use the notation $\mathbf{a}^*\mathbf{b}$ for their *scalar product* (the line vector is multiplied by the column vector) and $\mathbf{a}\mathbf{b}^*$ for their *tensor product* (the column vector is multiplied by the line vector). The product of a mapping A by a vector $\mathbf{a}$ will be denoted by $A < \mathbf{a} >$. The notation $\mathbf{b}^* A$ means the covector $\mathbf{c}^*$ defined by the rule $\mathbf{c}^* = (A^* < \mathbf{b} >)^*$. The divergence of a linear transformation A is the covector $divA$ such that, for any constant vector $\mathbf{a}$

$$div\, A\, \mathbf{a} \;=\; div\; (A \;< \mathbf{a} >).$$

The letter I will mean the identity transformation, and ∇ will mean the gradient line operator. The greek indices $\alpha, \beta = 1, 2$ will denote the parameters of components such as the densities ρ_α, the velocities $\mathbf{u}_\alpha$ etc.

2. Variational Principle

Let us suppose that a mixture of two miscible fluids is well described by the velocities $\mathbf{u}_1$, $\mathbf{u}_2$ of two components, the average densities ρ_1, ρ_2 and the total energy E. The total energy is divided into the kinetic energy T and the internal energy U. In the following, we will consider only mechanical processes by suppressing thermal evolution. Hence, U is purely mechanical part of the total internal energy. The kinetic energy, depending on the choice of a reference frame, is represented by the classic formula:

$$T = \frac{1}{2}\, \rho_1 |\mathbf{u}_1|^2 \; + \frac{1}{2}\, \rho_2\, |\mathbf{u}_2|^2.$$

The internal energy U is a Galilean invariant, it does not depend on the reference frame. Hence, U depends on ρ_1, ρ_2 and the relative velocity $\mathbf{w} = \mathbf{u}_2 - \mathbf{u}_1$.

Neglecting the dissipative effects, we propose the following extended form of Hamilton's principle of least action (Gavrilyuk *et al* [21]):

$$\delta \mathcal{I} = 0, \;\; \mathcal{I} = \int_{t_1}^{t_2}\!\!\int_{\mathcal{D}} L\; d\mathbf{x}dt, \qquad L \;=\; \rho_1\, \frac{|\mathbf{u}_1|^2}{2} + \rho_2\, \frac{|\mathbf{u}_2|^2}{2} \;- W(\rho_1, \rho_2, \mathbf{w}) \quad (2.1)$$

with additional kinematic contraints

$$\frac{\partial \rho_1}{\partial t} + div\; (\rho_1 \mathbf{u}_1) = 0\;,\;\; \frac{\partial \rho_2}{\partial t} + div\; (\rho_2 \mathbf{u}_2) = 0. \qquad (2.2)$$

Here $[t_1, t_2]$ is a time interval, $\mathcal{D}$ is a fixed bounded domain of the three-dimensional space with the boundary $\partial\mathcal{D}$. We will suppose that the slipping condition on $\partial\mathcal{D}$ is fulfilled for each component.

The internal energy U is the partial Legendre transformation of the potential $W(\rho_1, \rho_2, \mathbf{w})$ with respect to the variable $\mathbf{w}$:

$$U = W(\rho_1, \rho_2, \mathbf{w}) \; - \frac{\partial W}{\partial \mathbf{w}}\; \mathbf{w} = W + \mathbf{i}^*\; \mathbf{w}, \qquad \mathbf{i}^* = -\frac{\partial W}{\partial \mathbf{w}},$$

$$W = U - \mathbf{i}^*\mathbf{w}, \qquad \mathbf{w}^* = \frac{\partial U}{\partial \mathbf{i}}. \qquad (2.3)$$

The vector variable $\mathbf{i}$, which is also a Galilean invariant, can be called *the relative momentum.* Below, we will suppose that U is invariant under rotations (the case of *isotropic* media). Hence, U is a function of $i = |\; \mathbf{i}\; |$ and W is a function of $w = |\; \mathbf{w}\; |$. In this case the relations (2.3) can be represented in the form,

$$U = W - \frac{\partial W}{\partial w}\; w \;=\; W + i\; w,\;\; i \;=\; -\frac{\partial W}{\partial w},\;\; W = U - i\; w,\;\; w \;=\; \frac{\partial U}{\partial i}. \qquad (2.4)$$

If U does not depend on i, the variational principle (2.1), (2.2) coincides with classical Hamilton's principle of least action: the Lagrangian is the difference of the kinetic and the internal energy. The relation (2.3), (2.4) between U and W will be justified later when the conservation law of the total energy will be obtained.

3. Governing Equations and Jump Conditions

Let $\mathbf{x}$ be Eulerian coordinates, $\mathbf{X}_\alpha$ be Lagrangian coordinates of the α-th component, $\alpha = 1, 2$. The relation between Lagrangian and Eulerian coordinates is given by the diffeomorphisms of the domain $\mathcal{D}$ into $\mathcal{D}$:

$$\mathbf{x} = \boldsymbol{\phi}_\alpha\ (t, \mathbf{X}_\alpha),\ \mathbf{X}_\alpha = \boldsymbol{\psi}_\alpha\ (t, \mathbf{x}),\ \boldsymbol{\phi}_\alpha\ \circ\ \boldsymbol{\psi}_\alpha = I,\quad \alpha = 1, 2. \tag{3.1}$$

Let

$$F_\alpha = \frac{\partial\ \boldsymbol{\phi}_\alpha}{\partial\ \mathbf{X}_\alpha},\ \alpha = 1, 2 \tag{3.2}$$

be the deformation gradient at $\mathbf{X}_\alpha$ (or Jacobian matrix of the mapping $\boldsymbol{\phi}_\alpha$). Let us define the virtual motions of the mixture (Serrin [22], Gouin [23] and others):

$$\mathbf{x} = \mathbf{\Phi}_\alpha\ (t,\ \mathbf{X}_\alpha,\ \varepsilon_\alpha),\ \mathbf{X}_\alpha\ =\ \mathbf{\Psi}_\alpha\ (t,\ \mathbf{x},\ \varepsilon_\alpha),\ \mathbf{\Phi}_\alpha\ \circ\ \mathbf{\Psi}_\alpha\ =\ I, \tag{3.3}$$

where ε_α varies in the neighbourhood of zero. The real motion corresponds to $\varepsilon_\alpha = 0$:

$$\mathbf{\Phi}_\alpha\ (t,\ \mathbf{X}_\alpha, 0)\ =\ \boldsymbol{\phi}_\alpha\ (t,\ \mathbf{X}_\alpha),\ \mathbf{\Psi}_\alpha\ (t,\ \mathbf{x},\ 0)\ =\ \boldsymbol{\psi}_\alpha\ (t,\ \mathbf{x}).$$

The associated variations $\delta_\alpha \mathbf{x}$ and $\delta \mathbf{X}_\alpha$ are defined by the relations:

$$\delta_\alpha \mathbf{x}\ =\ \frac{\partial\ \mathbf{\Phi}_\alpha}{\partial\ \varepsilon_\alpha}\ (t, \mathbf{X}_\alpha, 0)\ ,\ \delta\ \mathbf{X}_\alpha\ =\ \frac{\partial\ \mathbf{\Psi}_\alpha}{\partial\ \varepsilon_\alpha}\ (t, \mathbf{x}, 0). \tag{3.4}$$

The definitions (3.1) - (3.4) imply that

$$\delta_\alpha \mathbf{x}\ =\ -F_\alpha < \delta\ \mathbf{X}_\alpha\ > . \tag{3.5}$$

Let

$$f_\alpha(t, \mathbf{x}),\quad \overset{\circ}{f}_\alpha\ (t, \mathbf{X}_\alpha)\ \equiv\ f_\alpha\ (t, \boldsymbol{\phi}_\alpha\ (t, \mathbf{X}_\alpha)\ ),$$

$$\hat{f}_\alpha\ (t, \mathbf{x}\ , \varepsilon_\alpha),\quad \tilde{f}_\alpha\ (t, \mathbf{X}_\alpha, \varepsilon_\alpha)\ \equiv \hat{f}_\alpha\ (t, \mathbf{\Phi}_\alpha\ (t, \mathbf{X}_\alpha,\ \varepsilon_\alpha), \varepsilon_\alpha\ )$$

be the unknown quantities of the α-th component (such as the density ρ_α, the velocity $\mathbf{u}_\alpha$ etc.) in Eulerian and Lagrangian coordinates and their perturbations, respectively. One defines Eulerian and Lagrangian variations of the variable f_α:

$$\delta\ f_\alpha\ =\ \frac{\partial \hat{f}_\alpha}{\partial\ \varepsilon_\alpha}\ (t, \mathbf{x}, 0)\ ,\ \delta\ \overset{\circ}{f}_\alpha =\ \frac{\partial \tilde{f}_\alpha}{\partial\ \varepsilon_\alpha}\ (t, \mathbf{X}_\alpha, 0).$$

It yields (Berdichevsky [17], Gouin [20,23])

$$\delta\ \overset{\circ}{f}_\alpha =\ \delta\ f_\alpha\ +\ \frac{\partial f_\alpha}{\partial\ \mathbf{x}}\ \ \delta_\alpha \mathbf{x}. \tag{3.6}$$

Using the Euler formula

$$\delta\ (det\ \overset{\circ}{F}_\alpha)\ =\ det\ \overset{\circ}{F}_\alpha\ \ div(\delta_\alpha\ \mathbf{x})\ ,\ div(\delta_\alpha\ \mathbf{x})\ =\ tr\left(\frac{\partial\ \delta_\alpha\ \mathbf{x}}{\delta\ \mathbf{x}}\right)$$

and the mass balance (2.2) in the form

$$\overset{\circ}{\rho}_\alpha\ \ det\ \overset{\circ}{F}_\alpha = \overset{\circ}{\rho}_\alpha\ \ (0, \mathbf{X}_\alpha),$$

we obtain Lagrangian variations of ρ_α:

$$\delta \overset{\circ}{\rho}_\alpha = - \overset{\circ}{\rho}_\alpha \quad div\ \delta_\alpha \mathbf{x}. \tag{3.7}$$

We also note that

$$\delta \overset{\circ}{\mathbf{u}}_\alpha = \frac{\partial}{\partial\ t}\ \delta_\alpha \mathbf{x}. \tag{3.8}$$

Using (3.6), (3.7), (3.8), we get

$$\delta\rho_\alpha = -\ div\ (\rho_\alpha\ \delta_\alpha \mathbf{x})\ ,\ \ \delta\ \mathbf{u}_\alpha = \frac{d_\alpha}{dt}\ \delta_\alpha\ \mathbf{x} - \frac{\partial \mathbf{u}_\alpha}{\partial\ \mathbf{x}} < \delta_\alpha \mathbf{x} >, \tag{3.9}$$

$$\frac{d_\alpha}{dt} = \frac{\partial}{\partial t} + \mathbf{u}_\alpha^*\ \nabla^*,$$

where ∇^* means the gradient column operator. We assume here that $\delta_\alpha \mathbf{x}$ are functions of Eulerian variables. We define the vectors

$$\mathbf{K}_\alpha^* \equiv \frac{1}{\rho_\alpha}\frac{\partial L}{\partial \mathbf{u}_\alpha} = \mathbf{u}_\alpha^* - (-1)^\alpha\ \frac{1}{\rho_\alpha}\ \frac{\partial W}{\partial w}\ \frac{\mathbf{w}^*}{w}, \tag{3.10}$$

where

$$L(\rho_1,\ \rho_2,\ \mathbf{u}_1,\ \mathbf{u}_2) = \frac{1}{2}\ \rho_1\ |\mathbf{u}_1|^2 + \frac{1}{2}\ \rho_2\ |\mathbf{u}_2|^2 - W\ (\rho_1, \rho_2, w)\ , w\ = |\ \mathbf{u}_2 - \mathbf{u}_1|.$$

Varying the variables $(\rho_1, \mathbf{u}_1)$ and $(\rho_2, \mathbf{u}_2)$ independently and denoting by $\delta_\alpha \mathcal{I}$ the corresponding variation of the functional $\mathcal{I}$, we obtain

$$\delta_\alpha \mathcal{I} = \int_{t_1}^{t_2} \int_{\mathcal{D}} \left(\delta\rho_\alpha \left(\frac{1}{2}|\mathbf{u}_\alpha|^2 - \frac{\partial W}{\partial \rho_\alpha} \right) + \rho_\alpha\ \mathbf{K}_\alpha^*\ \delta\ \mathbf{u}_\alpha \right) d\mathbf{x}\ dt.$$

Taking now into account the formulae (3.9), we get

$$\delta_\alpha \mathcal{I} = \int_{t_1}^{t_2} \int_{\mathcal{D}} \Bigg(-\ div\ (\rho_\alpha \delta_\alpha \mathbf{x}) \left(\frac{1}{2}\ |\mathbf{u}_\alpha|^2 - \frac{\partial W}{\partial \rho_\alpha} \right) +$$

$$+ \rho_\alpha \mathbf{K}_\alpha^* \left(\frac{d_\alpha}{dt}\ \delta_\alpha \mathbf{x} - \frac{\partial \mathbf{u}_\alpha}{\partial \mathbf{x}} < \delta_\alpha \mathbf{x} > \right)\Bigg)\ d\mathbf{x}\ dt\ =$$

$$= \int_{t_1}^{t_2} \int_{\mathcal{D}} \Bigg\{ -\rho_\alpha\ \delta_\alpha \mathbf{x}^* \Bigg(\frac{\partial \mathbf{K}_\alpha}{\partial t} + \frac{\partial \mathbf{K}_\alpha}{\partial \mathbf{x}} < \mathbf{u}_\alpha > + \frac{\partial \mathbf{u}_\alpha^*}{\partial \mathbf{x}} < \mathbf{K}_\alpha > +$$

$$+ \nabla^* \left(\frac{\partial W}{\partial \rho_\alpha} - \frac{|\mathbf{u}_\alpha|^2}{2} \right) \Bigg) + \frac{\partial}{\partial t}\ (\rho_\alpha \mathbf{K}_\alpha^* \delta_\alpha \mathbf{x}) +$$

$$+\ div \left(\left(\rho_\alpha \mathbf{u}_\alpha \mathbf{K}_\alpha^* + \rho_\alpha \left(\frac{\partial W}{\partial \rho_\alpha} - \frac{|\mathbf{u}_\alpha|^2}{2} \right) I \right) < \delta_\alpha \mathbf{x} > \right) \Bigg\}\ d\mathbf{x}\ dt\ = 0. \tag{3.11}$$

The mapping I means here the unit tensor and, as previously defined in the Introduction, for any vectors $\mathbf{a}$ and $\mathbf{b}$, $\mathbf{a}^*\ \mathbf{b}$ is the scalar product and $\mathbf{a}\ \mathbf{b}^*$ is the tensor product. If all the functions in (3.11) are smooth in the domain $\mathcal{D}$ and the variations $\delta_\alpha \mathbf{x}$ vanish on $\partial\mathcal{D}$, the divergence terms do not play any role. Therefore, we obtain the equations of motion:

$$\frac{\partial \mathbf{K}_\alpha}{\partial t} + \frac{\partial \mathbf{K}_\alpha}{\partial \mathbf{x}} < \mathbf{u}_\alpha > + \frac{\partial \mathbf{u}_\alpha^*}{\partial \mathbf{x}} < \mathbf{K}_\alpha > + \nabla^* \left(\frac{\partial W}{\partial \rho_\alpha} - \frac{|\mathbf{u}_\alpha|^2}{2} \right) = 0, \tag{3.12}$$

or, equivalently,

$$\frac{\partial \mathbf{K}_\alpha}{\partial t} + rot\, \mathbf{K}_\alpha \times \mathbf{u}_\alpha + \nabla^* \left(\frac{\partial W}{\partial \rho_\alpha} - \frac{|\mathbf{u}_\alpha|^2}{2} + \mathbf{K}_\alpha^* \, \mathbf{u}_\alpha \right) = 0. \qquad (3.12')$$

If the function $W\ (\rho_1,\ \rho_2,\ w)$ (or the internal energy $U(\rho_1,\ \rho_2,\ i)$) is given, the equations (3.10), (3.12) with the mass conservation laws (2.2) form the closed system of the governing equations. We will show in Section 6 that, under natural restrictions on the internal energy U (or W), the system is hyperbolic if the relative velocity is sufficiently small.

Now, suppose that the domain $\mathcal{D} \times [\ t_1, t_2\ ]$ is divided by a singular surface $S(t)$ having at any of its points the normal unit vector $\mathbf{n}$ and the normal speed of displacement D_n. Suppose also that at any point of $S(t)$ the right and left limits of $\mathbf{K}_\alpha$ and ρ_α exist, but not necessary equal. Then, the divergence terms in (3.11) give the jump conditions (the Rankine-Hugoniot conditions):

$$\Bigg[\mathbf{n}^* \left(\rho_\alpha\, \mathbf{u}_\alpha\, \mathbf{K}_\alpha^* + \rho_\alpha \left(\frac{\partial W}{\partial \rho_\alpha} - \frac{|\mathbf{u}_\alpha|^2}{2} \right) I \right) < \delta_\alpha \mathbf{x} > - \\ - D_n\, \rho_\alpha\, \mathbf{K}_\alpha^*\, \delta_\alpha \mathbf{x} \Bigg] = 0\ , \qquad (3.13)$$

where the square brackets denote the jump.

If the singular surface $S(t)$ is a *shock wave* for α-th constituent (i.e. $\mathbf{n}^*\, \mathbf{u}_\alpha - D_n \neq 0$ which means that the particles cross the surface), the formula (3.13) can be symplified. Indeed, using, (3.5) and the fact that $[\ \delta \mathbf{X}_\alpha\ ] = 0$, we obtain:

$$\left[\rho_\alpha\ (\ \mathbf{n}^*\ \mathbf{u}_\alpha - D_n) \mathbf{K}_\alpha^*\ F_\alpha + \rho_\alpha \left(\frac{\partial W}{\partial \rho_\alpha} - \frac{|\mathbf{u}_\alpha|^2}{2} \right) \mathbf{n}^* F_\alpha \right] = 0. \qquad (3.14)$$

The equations (3.14) contain not only $\mathbf{K}_\alpha$ and ρ_α but also the deformation gradient F_α. However, in the next section, we will obtain shock conditions in terms of $\mathbf{K}_\alpha$ and ρ_α, using a conservative form of the governing equations in Eulerian coordinates.

Finally note that the governing equations could be also obtained by using the method of the Lagrange multipliers.

4. Conservation Laws

Conservation laws, i.e. the expressions of the form

$$\frac{\partial\ P_0}{\partial t} + div\ \mathbf{P} = 0,$$

where P_0, $\mathbf{P}$ are functions of unknown variables, play an important role in the theory of hyperbolic equations (see, for instance, text-books by Serre [24] or Smoller [25]). The property of conservativeness of mathematical models, when the number of linear independent conservation laws admitted by the model is not less than the number of unknown variables, is necessary to determine weak solutions of the system. Some conservation laws can play a role of entropy, i.e. all admissible solutions of the system of conservation laws must satisfy the "*entropy inequality*"

$$\frac{\partial\ h_0}{\partial\ t} + div\ \mathbf{h} \leq 0\ ,$$

where h_0, $\mathbf{h}$ are functions of the unknown quantities.

The equations of motion (2.2), (3.12) admit two obvious additional conservation laws of the *total momentum* and the *total energy*, corresponding to the invariance of the Lagrangian

$$L = \frac{1}{2}\,\rho_1\,|\mathbf{u}_1|^2 \;+\; \frac{1}{2}\,\rho_2\,|\mathbf{u}_2|^2 \;-\; W\,(\rho_1\,,\;\rho_2\,,w)$$

with respect to time and space shifts. They can be obtained either using the theorem of E.Noether (Olver [26], Ovsyannikov [27]) or by direct calculations. The momentum conservation law is obtained multiplying the equations (3.12) by ρ_α and then summing:

$$\frac{\partial}{\partial\,t}\,(\,\rho_1\,\mathbf{u}_1^*\;+\;\rho_2\,\mathbf{u}_2^*)\;+\;div\,\Bigg(\,\rho_1\,\mathbf{u}_1\,\mathbf{u}_1^*\;+\rho_2\,\mathbf{u}_2\,\mathbf{u}_2^*\;-$$

$$-\;\frac{\partial W}{\partial w}\,\frac{\mathbf{w}\,\mathbf{w}^*}{w}\;+\;\left(\rho_1\,\frac{\partial W}{\partial \rho_1}\;+\;\rho_2\,\frac{\partial W}{\partial \rho_2}\;-\;W\right)I\,\Bigg) = 0. \tag{4.1}$$

It admits also an alternative form in terms of $\mathbf{K}_\alpha$ (see the definition (3.10)):

$$\frac{\partial}{\partial\,t}\,(\,\rho_1\,\mathbf{K}_1^*\;+\;\rho_2\,\mathbf{K}_2^*)\;+$$

$$+\;div\,\Bigg(\,\rho_1\,\mathbf{u}_1\,\mathbf{K}_1^*\;+\rho_2\,\mathbf{u}_2\,\mathbf{K}_2^*\;+\;\left(\rho_1\,\frac{\partial W}{\partial \rho_1}\;+\;\rho_2\,\frac{\partial W}{\partial \rho_2}\;-\;W\right)I\,\Bigg) = 0. \tag{4.1$'$}$$

Multiplying (3.12) by $\rho_\alpha\;\mathbf{u}_\alpha$ and then summing, we obtain the energy conservation law

$$\frac{\partial}{\partial\,t}\left(\rho_1\frac{|\mathbf{u}_1|^2}{2}\;+\;\rho_2\frac{|\mathbf{u}_2|^2}{2}\;+\;W\;-\;w\,\frac{\partial W}{\partial w}\right)\;+\;div\Bigg(\rho_1\mathbf{u}_1\left(\frac{|\mathbf{u}_1|^2}{2}\;+\;\frac{\partial W}{\partial \rho_1}\right)+$$

$$+\;\rho_2\,\mathbf{u}_1\left(\frac{|\mathbf{u}_2|^2}{2}\;+\;\frac{\partial W}{\partial \rho_2}\right)\;-\;\frac{\partial W}{\partial w}\left(\mathbf{u}_2\,\mathbf{u}_2^*\;-\;\mathbf{u}_1\,\mathbf{u}_1^*\right)\;<\;\frac{\mathbf{w}}{w}\;>\Bigg)\;=\;0. \tag{4.2}$$

It admits an alternative form:

$$\frac{\partial}{\partial\,t}\left(\rho_1\frac{|\mathbf{u}_1|^2}{2}\;+\;\rho_2\frac{|\mathbf{u}_2|^2}{2}\;+\;U\right)\;+\;div\,\Bigg(\rho_1\mathbf{u}_1\left(\frac{\partial W}{\partial \rho_1}\;-\;\frac{|\mathbf{u}_1|^2}{2}\;+\;\mathbf{K}_1^*\mathbf{u}_1\right)+$$

$$+\rho_2\,\mathbf{u}_2\left(\frac{\partial W}{\partial \rho_2}\;-\;\frac{|\mathbf{u}_2|^2}{2}\;+\;\mathbf{K}_2^*\mathbf{u}_2\right)\Bigg)\;=\;0. \tag{4.2$'$}$$

The energy conservation laws (4.2), (4.2') explains now the relation (2.4) between U and W:

$$U = W - w\,\frac{\partial W}{\partial w}.$$

It follows from (3.12'), that an additional conservation law can be obtained, if

$$rot\;\mathbf{K}_\alpha =\; 0. \tag{4.3}$$

In that case

$$\frac{\partial \mathbf{K}_\alpha}{\partial\,t}\;+\;\nabla^*\left(\mathbf{K}_\alpha^*\,\mathbf{u}_\alpha\;+\;\frac{\partial W}{\partial \rho_\alpha}\;-\;\frac{|\mathbf{u}_\alpha|^2}{2}\right)\;=\;0. \tag{4.4}$$

The equations (4.3), (4.4) are compatible. Indeed, the equation (4.4) admits a consequence

$$\frac{\partial}{\partial\,t}\,rot\;\mathbf{K}_\alpha =\; 0,$$

and hence, (4.3) can be considered as a restriction for the initial data.

To our knowledge, there is no additional conservation laws in terms of the variables ρ_α and $\mathbf{u}_\alpha$. Finally the system (2.2), (3.12) contains eight desired variables ρ_α, $\mathbf{u}_\alpha$, $\alpha = 1, 2$. If $rot\, \mathbf{K}_\alpha \neq 0$, it admits only six conservation laws (2.2), (4.1), (4.2). Hence, in the general case, the system seems not to be conservative.

Below, we will extend the set of desired variables, considering the deformation gradients F_α as the unknown quantities. We will show that the extended system is a system of conservation laws.

Straightforward calculations show that F_α satisfies the equation:

$$div \left(\frac{F_\alpha}{det\ F_\alpha} \right) = 0. \tag{4.5}$$

Using the Euler formula

$$\frac{d_\alpha}{dt} \left(det\ F_\alpha \right) = det\ F_\alpha\ div\ \mathbf{u}_\alpha, \qquad \frac{d_\alpha}{dt} = \frac{\partial}{\partial t} + \mathbf{u}_\alpha\ \nabla\ ,$$

we obtain

$$\frac{d_\alpha}{dt} \left(\frac{F_\alpha}{det\ F_\alpha} \right) = \left(\frac{\partial \mathbf{u}_\alpha}{\partial \mathbf{x}} - div\ \mathbf{u}_\alpha\ I \right) \frac{F_\alpha}{det\ F_\alpha}. \tag{4.6}$$

Taking into account (4.5), we rewrite (4.6) in the divergence form

$$\frac{\partial}{\partial t} \left(\frac{(F_\alpha)^m_s}{det\ F_\alpha} \right) + \frac{\partial}{\partial x^k} \left(\frac{(\mathbf{u}_\alpha)^k\ (F_\alpha)^m_s - (\mathbf{u}_\alpha)^m\ (F_\alpha)^k_s}{det\ F_\alpha} \right) = 0. \tag{4.7}$$

Here $(F_\alpha)^m_s$ are the components of F_α (m denotes the lines and s denotes the columns), $(\mathbf{u}_\alpha)^k$ are the components of $\mathbf{u}_\alpha$. The repeated latin indices imply summation. Conversely, applying the operator div to the equation (4.7), we obtain:

$$\frac{\partial}{\partial t}\, div \left(\frac{F_\alpha}{det\ F_\alpha} \right) = 0.$$

Hence, we can replace the equation (4.5) by the evolution equation (4.7), considering (4.5) as the restriction for initial data: if the condition (4.5) is fullfilled at $t = 0$, it is valid for any time. We note that the divergence form (4.7) was earlier obtained by Godunov and Romensky [28] in the theory of elasticity.

Finally, by using (4.7), we get the equation (3.12) in the conservative form:

$$\frac{\partial}{\partial t} \left(\frac{\mathbf{K}^*_\alpha\ F_\alpha}{det\ F_\alpha} \right) + div \left(\left(\mathbf{u}_\alpha\ \mathbf{K}^*_\alpha + \left(\frac{\partial W}{\partial \rho_\alpha} - \frac{|\mathbf{u}_\alpha|^2}{2} \right) I \right) \frac{F_\alpha}{det\ F_\alpha} \right) = 0, \tag{4.8}$$

which represents the conservation of *local momentum* of α-th phase. In the next section, we will show that the conservation law (4.8) corresponds to the jump conditions (3.13) obtained from the variational principle (2.1).

The equations (2.2), (4.5), (4.7), (4.8) are in the conservative form relative to the variables ρ_α, $\mathbf{u}_\alpha$ and F_α. It is also shown that the equations admit also conservation of total momentum, conservation of total energy and , in the case of $rot\, \mathbf{K}_\alpha = 0$, conservation of $\mathbf{K}_\alpha$.

5. Analysis of Rankine-Hugoniot Conditions

Let $S(t)$ be a singular surface with the unit normal vector $\mathbf{n}$ and the normal velocity D_n, where the functions ρ_α, $\mathbf{K}_\alpha$ and F_α have jumps. The equations (2.2), (4.7), (4.8) imply the following Rankine-Hugionot conditions:

$$[\ \rho_\alpha\ (\ \mathbf{n}^*\ \mathbf{u}_\alpha - D_n\ )\ ] = 0\ , \tag{5.1}$$

$$[\ (\ \mathbf{n}^*\ \mathbf{u}_\alpha - D_n\ )\ \frac{F_\alpha}{det\ F_\alpha}\ -\ \frac{\mathbf{u}_\alpha\ \ \mathbf{n}^*\ F_\alpha}{det\ F_\alpha}\ ]\ =\ 0\ , \tag{5.2}$$

$$\left[(\ \mathbf{n}^*\ \mathbf{u}_\alpha - D_n\ )\ \mathbf{K}^*_\alpha\ \frac{F_\alpha}{det\ F_\alpha}\ +\ \left(\frac{\partial W}{\partial \rho_\alpha}\ -\ \frac{|\mathbf{u}_\alpha|^2}{2}\right)\mathbf{n}^*\ \frac{F_\alpha}{det\ F_\alpha}\right]\ =\ 0. \tag{5.3}$$

If $\mathbf{n}^*\ \mathbf{u}_\alpha - D_n\ =\ 0$, we call it a *contact discontinuity.* If $\mathbf{n}^*\ \mathbf{u}_\alpha - D_n\ \neq\ 0$, we call it a *shock wave.*

Let us consider the case of contact discontinuity. It follows from (5.2) - (5.3) that

$$\left[\ \frac{\mathbf{u}_\alpha \mathbf{n}^*\ F_\alpha}{det\ F_\alpha}\right]\ =\ 0, \tag{5.2'}$$

$$\left[\ \left(\frac{\partial W}{\partial \rho_\alpha}\ -\ \frac{|\mathbf{u}_\alpha|^2}{2}\right)\frac{\mathbf{n}^*\ F_\alpha}{det\ F_\alpha}\ \right]\ =\ 0. \tag{5.3'}$$

Multiplying (5.2') by the normal vector $\mathbf{n}^*$ and taking into account that $[\ \mathbf{n}^*\mathbf{u}_\alpha\ ]\ =\ 0$, we get

$$\left[\ \frac{\mathbf{n}^*\ F_\alpha}{det\ F_\alpha}\ \right]\ =\ 0. \tag{5.2''}$$

The equations (5.2') and (5.2") imply that the velocity $\mathbf{u}_\alpha$ is continuous. Hence, it follows from (5.2'), (5.3') that the conditions on the contact discontinuity are given by

$$[\ \mathbf{u}_\alpha\ ]\ =\ 0,\ \ \left[\ \frac{\partial W}{\partial \rho_\alpha}\ \right]\ =\ 0. \tag{5.4}$$

Let us consider shock waves. First of all, we note that the condition (5.1) can be rewritten in the form

$$[\ \rho_\alpha\ det\ F_\alpha\ ]\ =\ 0. \tag{5.1'}$$

Hence, the relations (3.14) derived from the variational principle coincide with (5.3). As in the case of the contact discontinuity, we rewrite the equations (5.1) - (5.3) in terms of $\mathbf{u}_\alpha,\ \ \rho_\alpha$. Multiplying (5.2) by $\mathbf{n}^*$, we get the equation (5.2"). Further, let $S(t)$ be a shock surface in Eulerian coordinates, and $S_\alpha(t)$ be its image in Lagrangian coordinates of the $\alpha - th$ component. Let $\mathbf{q}_\alpha$ be a tangent vector to $S_\alpha(t)$. Then, $\mathbf{q}\ =\ F_\alpha < \mathbf{q}_\alpha >$ is a tangent vector to $S(t)$. Multiplying (5.3) from the right by $\mathbf{q}_\alpha$ and taking into account that $\mathbf{n}^*\ \mathbf{q}\ =\ 0$, we obtain:

$$\left[\ (\mathbf{n}^*\ \mathbf{u}_\alpha - D_n)\ \frac{\mathbf{K}^*_\alpha\ \mathbf{q}}{det\ F_\alpha}\ \right]\ =\ 0.$$

It follows then from the last relation and from equations (5.1), (5.1') that the tangential component $\mathbf{K}_{\alpha q}$ of $\mathbf{K}_\alpha\ =\ \mathbf{n}\ (\mathbf{n}^*\ \mathbf{K}_\alpha)\ +\ \mathbf{K}_{\alpha q},\ \ \mathbf{n}^*\ \mathbf{K}_{\alpha q}\ =\ 0$, is continuous:

$$[\ \mathbf{K}_{\alpha q}\ ] =\ 0. \tag{5.5}$$

Multiplying (5.2) by $\mathbf{K}^*_{\alpha q}$, we have:

$$\left[\ \mathbf{K}^*_{\alpha q}\ F_\alpha\ \ \frac{(\mathbf{n}^*\ \mathbf{u}_\alpha\ \ - D_n)}{det\ F_\alpha}\ \right]\ =\ \left[\ \frac{(\mathbf{K}^*_{\alpha q}\mathbf{u}_\alpha)\ \mathbf{n}^*\ F_\alpha}{det\ F_\alpha}\ \right]. \tag{5.6}$$

Replacing in (5.3) $\mathbf{K}_\alpha$ by $\mathbf{K}_{\alpha q}\ + \mathbf{n}\ (\mathbf{n}^*\ \mathbf{K}_\alpha)$, we get

$$\left[\ (\mathbf{n}^*\ \mathbf{u}_\alpha\ \ - D_n)\ \frac{(\mathbf{K}^*_{\alpha q} + \mathbf{n}^*\ (\mathbf{n}^*\ \mathbf{K}_\alpha)\ )\ F_\alpha}{det\ F_\alpha}\ + \right.$$

$$\left. +\ \left(\frac{\partial W}{\partial \rho_\alpha}\ -\ \frac{|\ \mathbf{u}_\alpha\ |^2}{2}\right)\frac{\mathbf{n}^*\ F_\alpha}{det\ F_\alpha}\ \right]\ =\ 0. \tag{5.7}$$

Equations (5.2"), (5.6), (5.7) imply then that

$$\left[\frac{\partial W}{\partial \rho_\alpha} - \frac{\mid \mathbf{u}_\alpha \mid^2}{2} + \mathbf{K}_\alpha^* \, \mathbf{u}_\alpha - D_n \, \mathbf{K}_\alpha^* \mathbf{n} \right] = 0. \tag{5.8}$$

Equations (5.1), (5.5), (5.8) are the Rankine-Hugionot conditions for shocks in terms of the variables $\mathbf{u}_\alpha, \rho_\alpha$. It is worth to note that the jump conditions (5.5), (5.8) for shocks coincide with the jump conditions for the equation (4.4). Nevertheless, we did not use in our derivation the hypothesis $rot \, \mathbf{K}_\alpha = 0$.

The conservation laws (4.1), (4.2) imply also additional jump conditions for the total momentum and energy. Hence, we obtain an *overdetermined* system of the Rankine-Hugoniot conditions in terms of the physical variables $\rho_\alpha, \; \mathbf{u}_\alpha, \; \alpha = 1, 2$. This is a consequence of the well-known fact that the same system of equations can be written in different divergence forms each of which defines different weak solution (see, e.g., [24], [25]). We can now question, which divergence form is the more appropriate? For one-velocity systems this choice is unambiguous. For example, for isentropic gas flows we use the conservation of mass and momentum. The mechanical energy plays the role of entropy: it decreases through the shocks [24]. The choice of appropriate shock conditions for the two-velocity case is less clear. Hamilton's principle provides a set of Rankine-Hugoniot conditions (3.14). As was shown in Section 4, these last conditions correspond to the divergence form (4.8) which represents the conservation of local momentum. Formally, equations (5.5), (5.8), which are issued from (3.14) and supplemented by the equations of mass conservation (5.1), form a complete set of Rankine-Hugoniot conditions. Similar to the one-velocity isentropic case, the energy conservation law (4.2) (or (4.2')) should apparently play for shocks the role of "entropy" inequality:

$$-D_n \, [\, E \,] + \left[\sum_{\alpha=1}^{2} \rho_\alpha \, \mathbf{n}^* \, \mathbf{u}_\alpha \left(\frac{\partial W}{\partial \rho_\alpha} - \frac{\mid \mathbf{u}_\alpha \mid^2}{2} + \mathbf{K}_\alpha^* \, \mathbf{u}_\alpha \right) \right] \leq 0,$$

$$E = \rho_1 \frac{|\mathbf{u}_1|^2}{2} + \rho_2 \frac{|\mathbf{u}_2|^2}{2} + U.$$

The jump conditions obtained are inconsistent with the conservation of the total momentum (4.1) (or (4.1')):

$$\left[\sum_{\alpha=1}^{2} \rho_\alpha (\mathbf{n}^* \, \mathbf{u}_\alpha - D_n) \, \mathbf{K}_\alpha^* + \mathbf{n}^* \left(\rho_1 \frac{\partial W}{\partial \rho_1} + \rho_2 \frac{\partial W}{\partial \rho_2} - W \right) \right] = 0.$$

Finally, we note that the system of the jump conditions for two-fluid models is generally underdetermined. This does not permit to define weak solutions. In our case, this system is overdetermined. Hamilton's principle provides a complete set of appropriate jump conditions. Is that choice correct? Only physical arguments are able to give a definite answer to this question.

6. Hyperbolicity

The property of hyperbolicity of governing equations is very important, because it implies the well-posedness of the Cauchy problem. Below, we will give a sufficient condition of the hyperbolicity of the system (2.2), (3.12) in the multi-dimensional case provided that $rot \, \mathbf{K}_\alpha = 0$.

First, we transform our system to a symmetric form. Considering the Lagrangian of our system

$$L = \sum_{\alpha=1}^{2} \frac{1}{2} \, \rho_\alpha \, |\mathbf{u}_\alpha|^2 - W \, (\rho_1 \, , \rho_2 \, , w),$$

we get:

$$dL = \sum_{\alpha=1}^{2} \left(\frac{\partial L}{\partial \rho_\alpha}\, d\rho_\alpha \;+\; \frac{\partial L}{\partial \mathbf{u}_\alpha}\; d\mathbf{u}_\alpha \right) \;=\; \sum_{\alpha=1}^{2} \left(\frac{\partial L}{\partial \rho_\alpha}\, d\rho_\alpha \;+\; \frac{1}{\rho_\alpha}\, \frac{\partial L}{\partial \mathbf{u}_\alpha}\, \rho_\alpha\, d\mathbf{u}_\alpha \right) \;=$$

$$= \sum_{\alpha=1}^{2} \left(\frac{\partial L}{\partial \rho_\alpha} d\rho_\alpha \;+\; \mathbf{K}_\alpha^* \,\left(d\,(\rho_\alpha\, \mathbf{u}_\alpha) - \mathbf{u}_\alpha d\, \rho_\alpha\,\right) \right) \;=$$

$$= \sum_{\alpha=1}^{2} \left(\left(\frac{\partial L}{\partial \rho_\alpha} \; - \mathbf{K}_\alpha^*\, \mathbf{u}_\alpha \right) d\, \rho_\alpha \;+\; \mathbf{K}_\alpha^*\, d(\rho_\alpha\, \mathbf{u}_\alpha) \right) \;=\; \sum_{\alpha=1}^{2} \left(\sigma_\alpha\, d\rho_\alpha + \mathbf{K}_\alpha^*\, d\mathbf{j}_\alpha\,\right),$$

where

$$\sigma_\alpha \;= \frac{\partial L}{\partial \rho_\alpha} \;-\; \mathbf{K}_\alpha^*\, \mathbf{u}_\alpha \;=\; -\left(\frac{\partial W}{\partial \rho_\alpha} \;-\frac{1}{2}\, |\mathbf{u}_\alpha|^2 \;+\; \mathbf{K}_\alpha^*\, \mathbf{u}_\alpha \right),\; \mathbf{j}_\alpha \;=\; \rho_\alpha \mathbf{u}_\alpha.$$

Or, equivalently,

$$d\,(L - \sum_{\alpha=1}^{2} \sigma_\alpha\, \rho_\alpha) \;=\; \sum_{\alpha=1}^{2} -\,\rho_\alpha d\sigma_\alpha \;+\; \mathbf{K}_\alpha^*\, d\mathbf{j}_\alpha. \tag{6.1}$$

Let us introduce

$$G(\sigma_1,\, \sigma_2,\, \mathbf{j}_1,\, \mathbf{j}_2) \;=\; L(\rho_1,\, \rho_2,\, \mathbf{j}_1,\, \mathbf{j}_2) \;-\; \sum_{\alpha=1}^{2} \sigma_\alpha\, \rho_\alpha \;=\; L \;-\; \sum_{\alpha=1}^{2} \frac{\partial L}{\partial \rho_\alpha}\, \rho_\alpha. \tag{6.2}$$

The function G is a partial Legendre transformation of $L\,(\rho_1,\, \rho_2,\, \mathbf{j}_1,\, \mathbf{j}_2)$ with respect to the variables ρ_α:

$$\frac{\partial G}{\partial \sigma_\alpha} \;=\; -\,\rho_\alpha, \qquad \frac{\partial G}{\partial \mathbf{j}_\alpha} \;=\; \mathbf{K}_\alpha^*. \tag{6.3}$$

By using (6.1) - (6.3) we get:

$$\frac{\partial}{\partial t}\left(\frac{\partial G}{\partial \sigma_\alpha} \right) \;-\; div\; \mathbf{j}_\alpha \;=\; 0,$$

$$\frac{\partial}{\partial t}\left(\frac{\partial G}{\partial \mathbf{j}_\alpha} \right) \;-\; \nabla\, \sigma_\alpha \;=\; 0.$$

Or

$$\frac{\partial}{\partial t}\left(\frac{\partial G}{\partial \sigma_\alpha} \right) \;-\; div\, \left(\frac{\partial}{\partial \sigma_\alpha} \left(\sum_{\beta=1}^{2} \sigma_\beta\, \mathbf{j}_\beta \right) \right) \;=\; 0, \tag{6.4}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial G}{\partial \mathbf{j}_\alpha} \right) \;-\; div\, \left(\frac{\partial}{\partial \mathbf{j}_\alpha} \left(\sum_{\beta=1}^{2} \sigma_\beta\, \mathbf{j}_\beta \right) \right) \;=\; 0. \tag{6.5}$$

The system (6.4), (6.5) can be rewritten in a symmetric form (Friedrichs [29], Friedrichs & Lax [30], Godunov [31], Godunov & Romensky [28]):

$$A\, \frac{\partial \mathbf{u}}{\partial t} \;+\; B^i\, \frac{\partial \mathbf{u}}{\partial x^i} \;=\; 0 \;\;, \quad A = A^*\;, B^i \;= (B^i)^*, \quad i = 1,\, 2,\, 3, \tag{6.6}$$

where

$$\mathbf{u}^* = (\sigma_1 , \sigma_2 , \mathbf{j}_1^*, \mathbf{j}_2^*) , \qquad A = \frac{\partial^2 G}{\partial \mathbf{u}^2}$$

and the matrices B^i can be obtained from (6.4), (6.5). If, moreover, the matrix A is positive, the system (6.6) is hyperbolic. It is worth to note that equations (6.4), (6.5) admit a conservation law of the form

$$\frac{\partial}{\partial t}\left(\sum_{\alpha=1}^{2} \sigma_\alpha \frac{\partial G}{\partial \sigma_\alpha} + \frac{\partial G}{\partial \mathbf{j}_\alpha} \mathbf{j}_\alpha - G\right) - div \left(\sum_{\alpha=1}^{2} \sigma_\alpha \mathbf{j}_\alpha \right) = 0,$$

which coincides with the equation of energy (4.2'). (see also Godunov [31]).

So, we need to prove convexity of G $(\sigma_1, \sigma_2, \mathbf{j}_1, \mathbf{j}_2)$. As we have previously mentioned, G is the Legendre transformation of L $(\rho_1, \rho_2, \mathbf{j}_1, \mathbf{j}_2)$ with respect to ρ_1, ρ_2 (see the formulae (6.2) , (6.3)). If L is a *convex* function with respect to $\mathbf{j}_1$, $\mathbf{j}_2$ and *concave* with respect to ρ_1, ρ_2, then G $(\sigma_1, \sigma_2, \mathbf{j}_1, \mathbf{j}_2)$ will be convex, that means the hyperbolicity of our system. Hence, it is sufficient to prove that the symmetric matrices $L_{\mathbf{jj}}$ and $L_{\rho\rho}$, defined below, are positive and negative definite, respectively:

$$L_{\mathbf{jj}} \equiv \begin{pmatrix} L_{\mathbf{j}_1\mathbf{j}_1} & L_{\mathbf{j}_1\mathbf{j}_2} \\ L_{\mathbf{j}_1\mathbf{j}_2} & L_{\mathbf{j}_2\mathbf{j}_2} \end{pmatrix} > 0,$$

$$L_{\rho\rho} \equiv \begin{pmatrix} L_{\rho_1\rho_1} & L_{\rho_1\rho_2} \\ L_{\rho_1\rho_2} & L_{\rho_1\rho_2} \end{pmatrix} < 0.$$

It follows from (1.2) that for the isotropic case

$$L (\rho_1, \rho_2, \mathbf{j}_1, \mathbf{j}_2) = \frac{|\mathbf{j}_1|^2}{2\rho_1} + \frac{|\mathbf{j}_2|^2}{2\rho_2} - W\left(\rho_1, \rho_2, \left| \frac{\mathbf{j}_2}{\rho_2} - \frac{\mathbf{j}_1}{\rho_1} \right| \right).$$

Then,

$$L_{\mathbf{j}_\alpha} = \frac{{\mathbf{j}_\alpha}^*}{\rho_\alpha} - (-1)^\alpha \frac{1}{\rho_\alpha} \frac{\partial W}{\partial w} \frac{\mathbf{w}^*}{w},$$

$$L_{\mathbf{j}_1\mathbf{j}_2} = \frac{1}{\rho_1\rho_2} \frac{\partial^2 W}{\partial w^2} \frac{\mathbf{w}\mathbf{w}^*}{w^2},$$

$$L_{\mathbf{j}_\alpha\mathbf{j}_\alpha} = \frac{1}{\rho_\alpha} I - \frac{1}{\rho_\alpha^2} \frac{\partial^2 W}{\partial w^2} \frac{\mathbf{w}\mathbf{w}^*}{w^2}.$$

Straightforward calculations shows that $L_{\mathbf{jj}} > 0$, if $\dfrac{\partial^2 W}{\partial w^2} < 0$.

Now, we calculate $L_{\rho\rho}$:

$$\frac{\partial L}{\partial \rho_\alpha} = - \frac{|\mathbf{j}_\alpha|^2}{2{\rho_\alpha}^2} - \frac{\partial W}{\partial \rho_\alpha} + (-1)^\alpha \frac{\partial W}{\partial w} \frac{\mathbf{w}^*\mathbf{j}_\alpha}{w\rho_\alpha^2},$$

$$\frac{\partial^2 L}{\partial \rho_1 \partial \rho_2} = - \frac{\partial^2 W}{\partial \rho_1 \partial \rho_2} - \frac{\partial}{\partial \rho_2}\left(\frac{\partial W}{\partial w} \frac{\mathbf{w}^*\mathbf{j}_1}{w\rho_1^2} \right),$$

$$\frac{\partial^2 L}{\partial \rho_\alpha^2} = \frac{|\mathbf{j}_\alpha|^2}{\rho_\alpha^3} - \frac{\partial^2 W}{\partial \rho_\alpha^2} - \frac{\partial}{\partial \rho_\alpha}\left(\frac{\partial W}{\partial w} \frac{\mathbf{w}^*\mathbf{j}_\alpha}{w\rho_\alpha^2} \right).$$

Consequently, $L_{\rho\rho} < 0$ if the velocities $\mathbf{u}_\alpha$ are sufficiently small, and the function W is convex with respect to ρ_1, ρ_2. However, the governing equations are invariant under the Galilean group of transformations

$$\mathbf{x}' = \mathbf{x} + \mathbf{U}t, \; \mathbf{u}_\alpha' = \mathbf{u}_\alpha + \mathbf{U}, \; t' = t.$$

That means that the condition "the velocities $\mathbf{u}_\alpha$ are sufficiently small" can be replaced by "the relative velocity $\mathbf{w}$ is sufficiently small". Hence, the conditions

$$\frac{\partial^2 W}{\partial w^2} < 0 \ , \ \frac{\partial^2 W}{\partial \rho_1^2} > 0, \ \frac{\partial^2 W}{\partial \rho_1^2}\frac{\partial^2 W}{\partial \rho_2^2} - \left(\frac{\partial^2 W}{\partial \rho_1 \partial \rho_2}\right)^2 > 0 \qquad (6.7)$$

guarantee the hyperbolicity of our system for small relative velocity of phases. Due to (2.4), the inequalities (6.7) mean the convexity of the internal energy $U(\rho_1, \rho_2, i)$, that corresponds to a natural condition of *thermodynamic stability.*

Finally, we have established that the thermodynamic stability implies the hyperbolicity of the governing equations for small relative velocity $\mathbf{w}$, provided that $rot\ \mathbf{K}_i = 0$. The last condition is always fullfilled for one-dimensional flows.

Acknowledgements

The authors thank D. Serre for helpful discussions.

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